PROJECTIVE CONFIGURATIONS

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Materials:
Painted wood and wire

Projective geometry is an elegant and enlightening domain of spatial thinking and doing. As a branch of mathematics, it is fairly unknown but intuitively accessible to a general audience. Projective geometry grew out of the linear perspective studies of Renaissance painters, although essentially projective results can be found already from Apollonius of Perga (c. 200 BC) and Pappus of Alexandria (c. 300 AD).

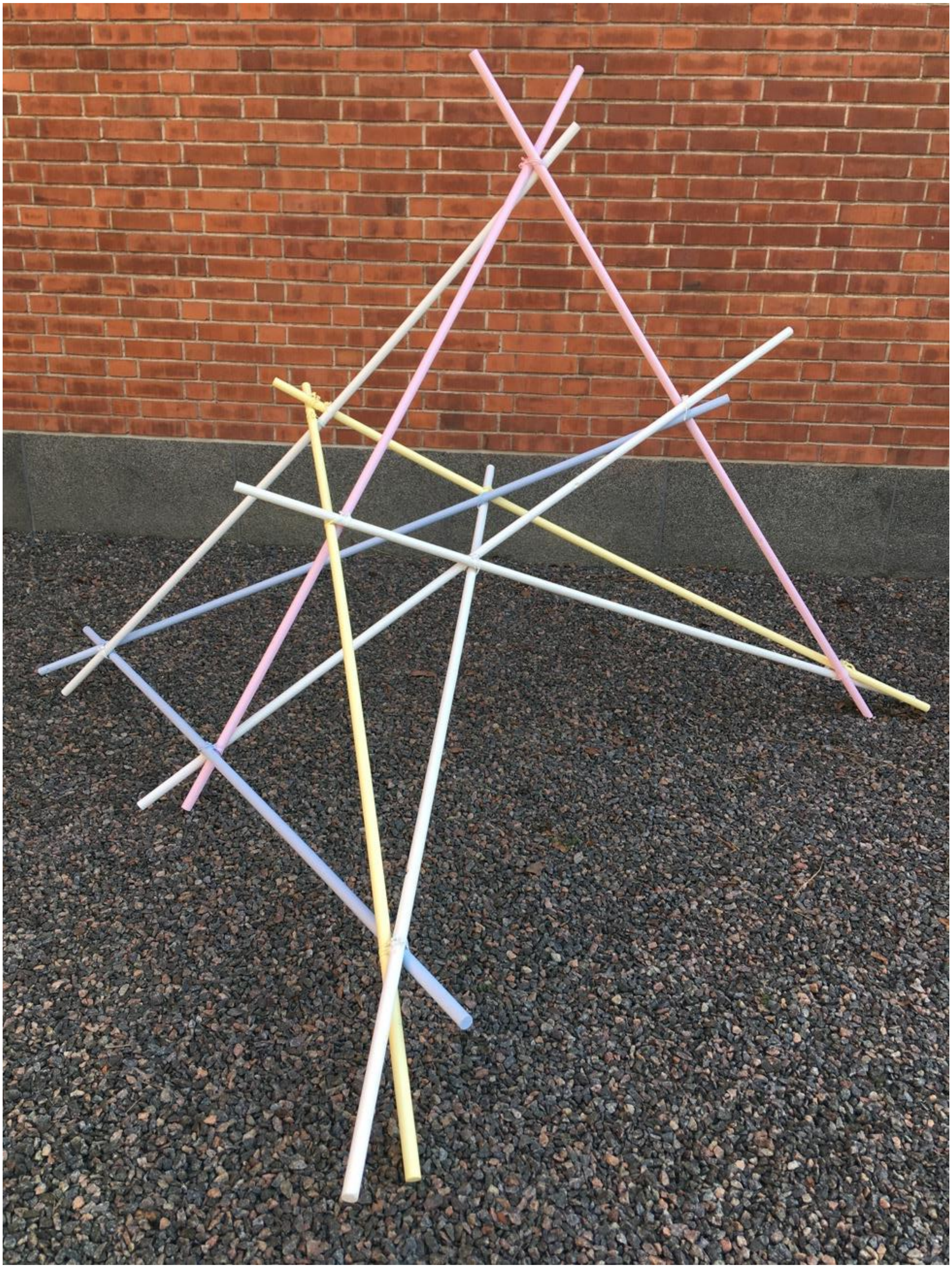
Projective geometry differs from the ordinary school geometry in the sense that it does not have the notion of parallelism. This means that in projective plane any pair of lines, and in projective space any pair of planes, will intersect each other. Another difference between school geometry and the projective one is that the latter does not have a principle of metric measurement, and concepts such as lengths, angles, areas, or volumes do not consequently belong to its vocabulary. Instead, projective geometry studies those characteristics of shapes and figures that remain unchanged regardless of the chosen point of view, such as the concurrency of lines and the collinearity of points.

The figures and transformations of projective geometry can be explored through configurations – structures consisting of points, lines, and planes. Although the configurations reside naturally in projective space, they can be modelled by finite sticks in our locally-Euclidean physical space. During a series of online lectures and hand-on workshops, the students were introduced to the basics of projective geometry. Before participating the workshop on-site, students investigated different projective configurations and the phenomena connected to them by assembling miniature models independently from bamboo sticks.

The five designs displayed arose from Luotoniemi's artistic research and were assembled with students from wooden rods during an outdoors workshop. The activity provided an instructive, simple, and embodied method to study mathematically challenging phenomena. The models illuminate new interdisciplinary connections between arts and mathematics, and facilitate intuitive, playfully experimental and imaginative approach to visual arts topics such as graphical perspective and sculptural composition.

To see videos of miniature configuration building, see:

Complete tetrahedron: https://youtu.be/fby53U_n4o8
Desargues' configuration: https://youtu.be/lOkbYVcRbQc
Complete hexahedron: https://youtu.be/9lkI3_1CZ2M
Reye's configuration: https://youtu.be/ZVtHbLF2G9M



Desargues' Configuration

DESARGUES' CONFIGURATION

Originally constructed to relate a pair of triangles in the plane, the Desargues' theorem states that two triangles are in perspective from a point if and only if they are also in perspective from a line. That is, if the lines joining the corresponding vertices are concurrent (at the center of perspectivity), the intersection points of the corresponding sides are collinear (along the axis of perspectivity). By observing the situation in three-dimensional projective space, the triangles can be thought of as cross-sections of a three-sided cone with an apex at the center of perspectivity. The axis of perspectivity is then revealed to be simply the line of intersection of the planes of the triangles.

Building a stick model of the Desargues' configuration amounts to introducing a fifth plane into the complete tetrahedron. Five planes of projective 3-space in general position intersect each other in the same pattern of ten lines (with three points along each) and ten points (with three lines through each). The planes partition the space into five tetrahedral cells, and ten triangular prisms. Two tetrahedra and two prisms appear intact in the stick model, all the other cells are split in two by the plane at infinity. The colors of the model illuminate one pair of perspective triangles having corresponding sides painted with blue, pink, and yellow. White rods joining the corresponding vertices are concurrent at the center of perspectivity, and the axis of perspectivity is another white rod.

The Desargues' configuration has also a connection to four-dimensional geometry, as it can be interpreted as a gnomonic projection of a four-dimensional solid called the expanded pentachoron.

Desargues' Configuration – a view showing another pair of triangles and their center of perspectivity

Complete Hexahedron

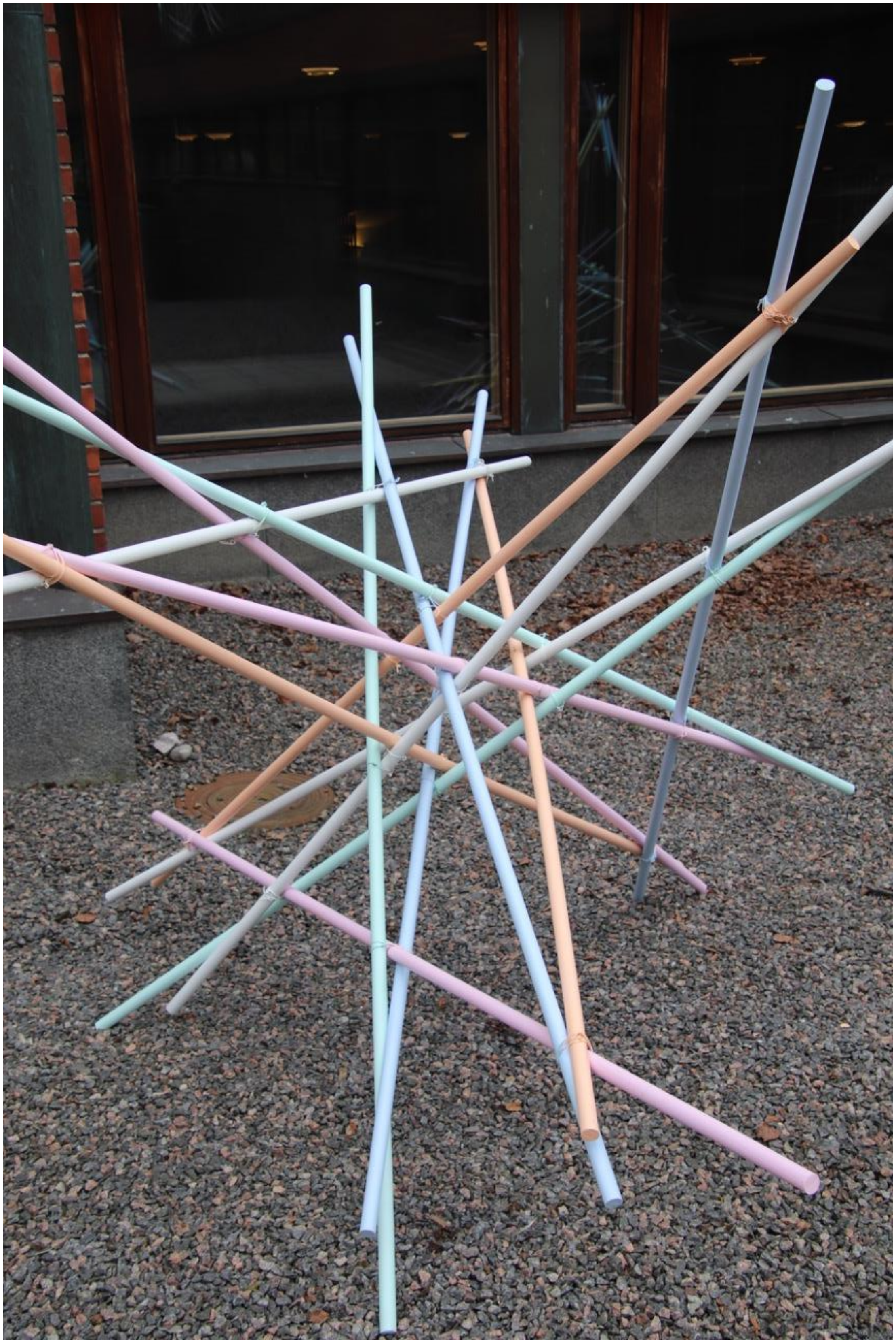
COMPLETE HEXAHEDRON

Selecting the layout for a stick model of a configuration amounts to adjusting how the plane at infinity is situated with respect to the configuration portrayed. This can be illustrated by adding a sixth plane into the Desargues' configuration. The reason why the Desargues' configuration in 3D always has the same layout, is that six planes in general position will intersect each other in projective space in the same configuration of fifteen lines (with four points along each) and twenty points (with three lines through each). The complete hexahedron partitions the space into six tetrahedral cells, twelve triangular prisms, and two cuboids. There are also six cells bounded by two triangles, two quadrilaterals, and two pentagons each.

As the plane at infinity can be positioned in several distinct fashions with respect to the six planes of the configuration, there exist different layouts for the stick figure. The physical model portrays a layout having a three-fold rotational symmetry around its vertical axis. In this layout three tetrahedra, six prisms, and one cuboid appear intact. The rest of the cells can again be found by following the edges sticking outward and tracing them around the cyclic lines to the opposite side of the configuration.

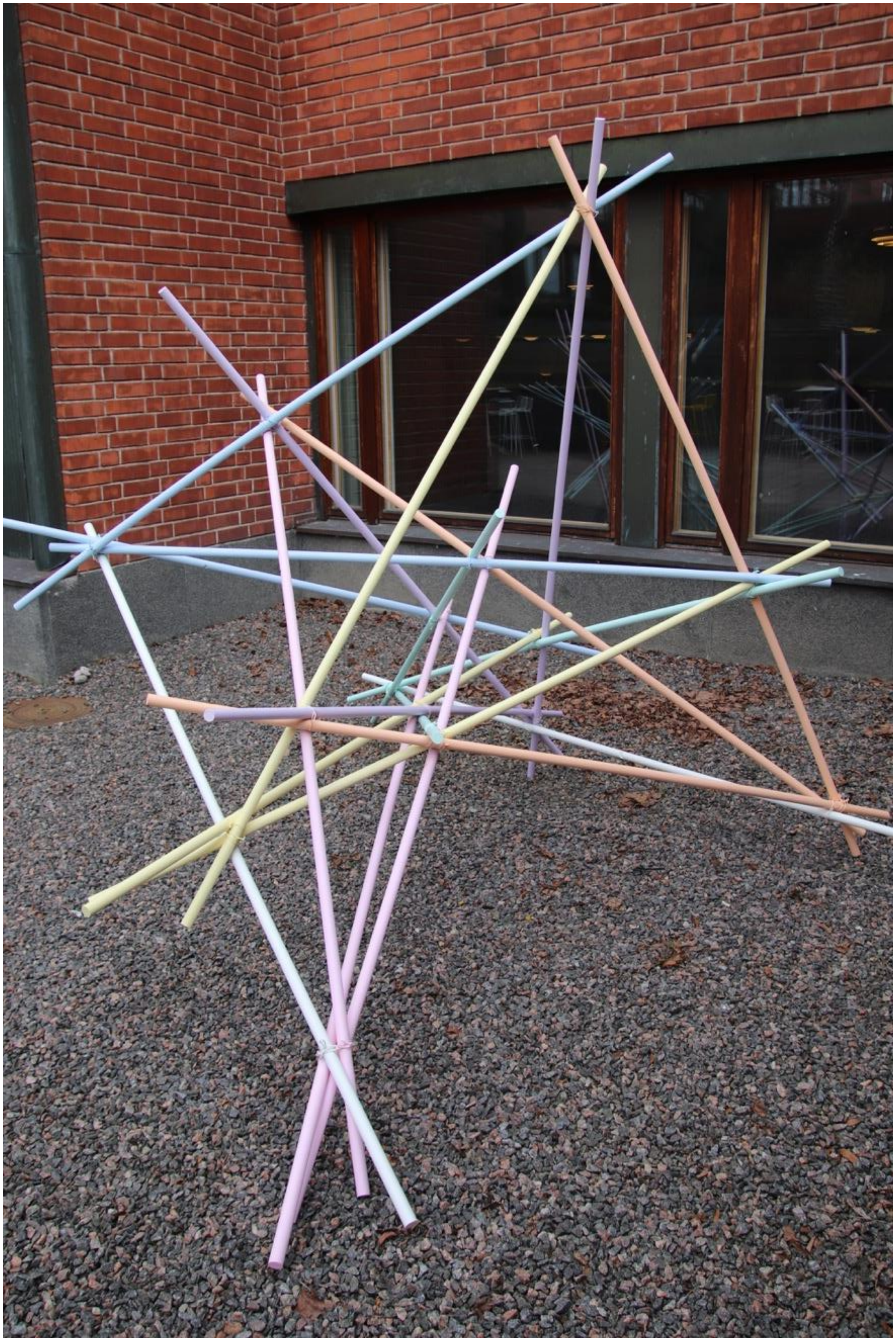
The theorem embodied by the configuration states that two quadrilaterals are in perspective from a line (i.e., the intersection points of the corresponding sides are collinear) if and only if the lines determined by their corresponding vertices form a quadrangle. By observing the situation in three-dimensional space, the quadrilaterals can be thought of as cross sections of a complete tetrahedron, the axis of perspectivity being the intersection line of their planes.

The model is painted with five colors (blue, pink, white, green, and orange) so that each color appears exactly once along each plane.



Complete Hexahedron – a view revealing the three-fold rotational symmetry

Complete Hexahedron – a view of the intact cuboid-shaped cell



Complete Hexachoron

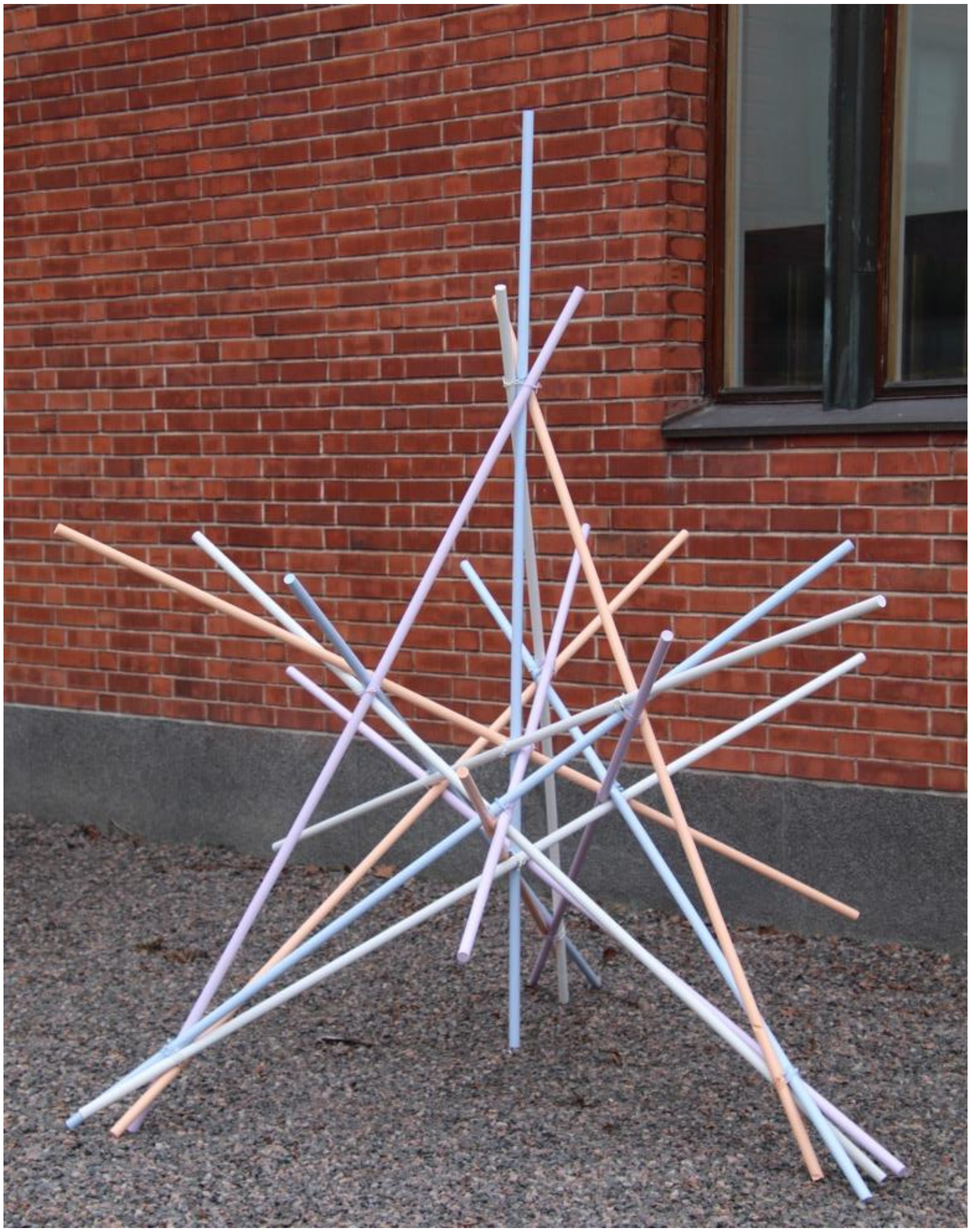
COMPLETE HEXACHORON

A plane dual of the quadrilateral theorem – illustrated by the complete hexahedron, can be formulated as: two quadrangles are in perspective from a point if and only if the meets of their corresponding edges are the vertices of a quadrilateral. In three-dimensional projective space this fact becomes a generalization of the Desargues' theorem, stating that two tetrahedra are in perspective from a point if and only if they are also in perspective from a plane (i.e., the intersection points of the corresponding edges are coplanar). Observed in a four-dimensional projective space, the theorem becomes almost self-evident. The pair of tetrahedra can be understood as two cross-sections through a bundle of four hyperplanes, concurrent at the center of perspectivity. The plane where the corresponding edges of the tetrahedra meet is then the plane of intersection for the hyperplanes of the tetrahedra.

We may call this configuration a *complete hexachoron*, as it consists of six hyperplanes in general position. The hyperplanes intersect each other in fifteen planes, twenty lines, and fifteen points. Each of its planes goes through six points and four lines, whereas each line goes through three points, and is an intersection of three planes. Each of the points of the configuration has four lines and six planes going through it. There are six instances of the Desargues' configuration present in the configuration, corresponding to the six hyperplanes.

Finding the shapes bounded by the hyperplanes requires understanding of not just projective geometry, but also of four-dimensional space. The configuration can be interpreted as a gnomonic projection of five-dimensional solid called the expanded 5-simplex. This polytope has three kinds of four-dimensional hypercells – pentachora, tetrahedral prisms, and 3–3 duoprisms. The coloring of the model illuminates a single 3–3 duoprismatic cell, with its six facets colored blue, pink, yellow, orange, green, and purple. The vanishing points where these lines meet lie on two horizon lines, illustrated by white rods.

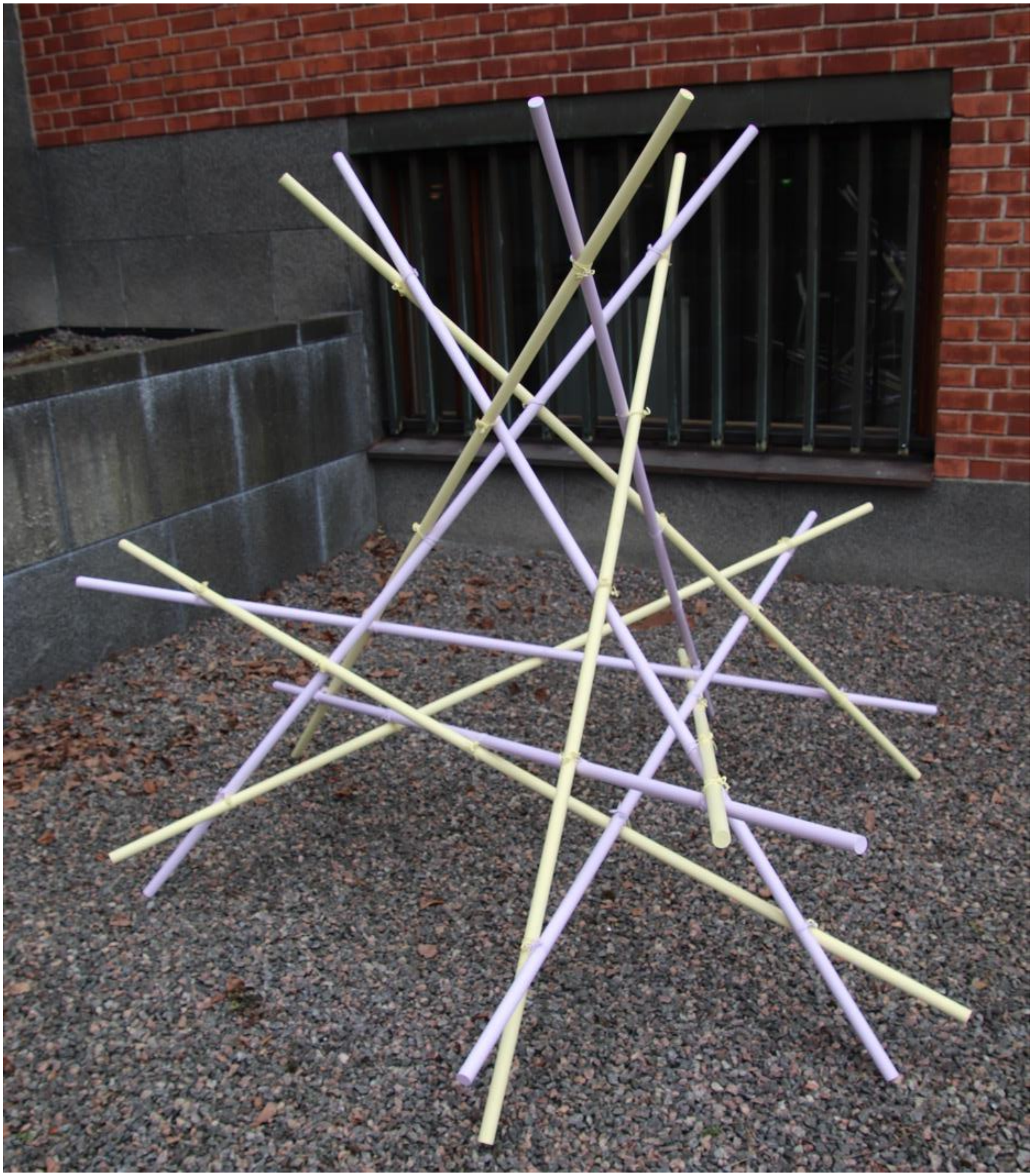

Rods for the *Complete Hexachoron* painted and ready for assembling



Reye's Configuration

REYE'S CONFIGURATION

When a four-dimensional solid called the 24-cell, bounded by twenty-four octahedra, is gnomonically projected into our three-dimensional space, is yields a projective figure called Reye's configuration. This structure consists of twelve points, sixteen lines, and twelve planes. Each of its points has four lines and six planes through it, each plane goes through six points and four lines, and each of its lines is incident with three points and three planes. The regularity of the configuration allows a four-coloring of the lines (blue, orange, white, pink), where none of the colors touch each other, but each of them appears at every point and in every plane.



Schläfli's Double-Six

SCHLÄFLI'S DOUBLE–SIX

Another interesting subject for a stick model portrayal is the Schläfli's double-six configuration, consisting of two sets (yellow and purple in the model) of six skew lines in projective space. Although the lines belonging to the same set are mutually skew, each of line of one set intersects five lines of the other set, amounting to thirty points in total. The lines of the configuration actually determine a cubic surface, which has not just twelve but twenty-seven straight lines.